RAN-REURINGS THEOREMS IN ORDERED METRIC SPACES

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Abstract. The Ran-Reurings fixed point theorem [Proc. Amer. Math. Soc., 132 (2004), 1435-1443] is but a particular case of Maia's [Rend. Sem. Mat. Univ. Padova, 40 (1968), 139-143]. A functional version of this last result is then provided, in a convergence-metric setting.

1. Introduction

Let X be a nonempty set. Take a metric $d(.,.)$ over it; as well as a self-map $T : X \to X$. We say that $x \in X$ is a *Picard point* (modulo (d,T)) if **i)** $(T^n x; n \geq 0)$ (=the *orbit* of x) is d-convergent, **ii)** $z := \lim_n T^n x$ is in Fix(T) (i.e., $z = Tz$). If this happens for each $x \in X$ and **iii)** Fix(T) is a singleton, then T is referred to as a *Picard operator* (modulo d); cf. Rus [23, Ch 2, Sect 2.2]. For example, such a property holds whenever d is *complete* and T is *d-contractive*; cf. (b04). A structural extension of this fact – when an *order* $(\leq)$ on X is being added – was obtained in 2004 by Ran and Reurings [21]. For each $x, y \in X$, denote

(a01) $x <> y$ iff either $x \leq y$ or $y \leq x$ (i.e.: x and y are comparable).

This relation is reflexive and symmetric; but not in general transitive. Call the self-map T, *$(d, \leq; \alpha)$-contractive* (for $\alpha > 0$), if

(a02) $d(Tx, Ty) \leq \alpha d(x, y)$, $\forall x, y \in X$, $x \leq y$.

If this holds for some $\alpha \in]0, 1[$, we say that T is *$(d, \leq)$-contractive*.

Theorem 1. *Let d be complete and T be d-continuous. In addition, assume that T is $(d, \leq)$-contractive and*

(a03) $X(T, <>) := \{x \in X; x <> Tx\}$ *is nonempty*

(a04) *T is monotone (increasing or decreasing)*

(a05) *for each $x, y \in X$, $\{x, y\}$ has lower and upper bounds.*

Then, T is a Picard operator (modulo d).

According to many authors (cf. [1], [4], [8], [18], [19] and the references therein), this result is credited to be the first extension of the classical 1922 Banach's contraction mapping principle [2] to the realm of (partially) ordered metric spaces. Unfortunately, the assertion is not true: some early statements of this type have been obtained two decades ago by Turinici [27], in the context of quasi-ordered metric spaces. (We refer to Section 5 below for details).

Now, the Ran-Reurings fixed point result found some useful applications to matrix and differential/integral equations. So, it cannot be surprising that, soon after, many extensions of Theorem 1 were provided; see the quoted papers for details. It

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is therefore natural to discuss the position of Theorem 1 within the classification scheme proposed by Rhoades [22]. The conclusion to be derived reads (cf. Section 2): the Ran-Reurings theorem is but a particular case of the 1968 fixed point statement in Maia [15, Theorem 1]. Further, in Section 3, some extensions are given for this last result, in the context of quasi-ordered convergence almost metric spaces. Some trivial quasi-order variants of these are then discussed in Section 4; note that, as a consequence of this, one gets the related contributions in the area due Kasahara [12] and Jachymski [10], as well as the order type statement in O'Regan and Petruşel [19]. Some other aspects will be delineated elsewhere.

2. Main result

Let $(X, d; \leq)$ be an ordered metric space; and $T : X \to X$, a self-map of X. Given $x, y \in X$, any subset $\{z_1, ..., z_k\}$ (for $k \geq 2$) in X with $z_1 = x$, $z_k = y$, and $[z_i <> z_{i+1},\ i \in \{1, ..., k-1\}]$ will be referred to as a $<>$-*chain* between x and y; the class of all these will be denoted as $C(x, y; <>)$. Let $\sim$ stand for the relation over X attached to $<>$ as

(b01) $x \sim y$ iff $C(x, y; <>)$ is nonempty.

Clearly, $(\sim)$ is reflexive and symmetric; because so is $<>$. Moreover, $(\sim)$ is transitive; hence, it is an equivalence over X.

The following variant of Theorem 1 is our starting point.

Theorem 2. *Let d be complete and T be d-continuous. In addition, assume that T is $(d, \leq)$-contractive and*

(b02) *T is $<>$-increasing [$x <> y$ implies $Tz <> Ty$]*

(b03) *$(\sim) = X \times X$ [$C(x, y; <>)$ is nonempty, for each $x, y \in X$].*

Then, T is a Picard operator (modulo d).

This result includes Theorem 1; because (a04) $\Longrightarrow$ (b02), (a05) $\Longrightarrow$ (b03). [For, given $x, y \in X$, there exist, by (a05), some $u, v \in X$ with $u \leq x \leq v$, $u \leq y \leq v$. This yields $x <> u$, $u <> y$; wherefrom, $x \sim y$]. In addition, it tells us that the regularity condition (a03) is superfluous.

The remarkable fact to be noted is that Theorem 2 (hence the Ran-Reurings statement as well) is deductible from the Maia's fixed point statement [15, Theorem 1]. Let $e(.,.)$ be another metric over X. Call $T : X \to X$, $(e; \alpha)$-*contractive* (for $\alpha > 0$) when

(b04) $e(Tx, Ty) \leq \alpha e(x, y)$, $\forall x, y \in X$;

if this holds for some $\alpha \in]0, 1[$, the resulting convention will read as: T is e-*contractive*. Further, let us say that d is *subordinated* to e when $d(x, y) \leq e(x, y)$, $\forall x, y \in X$. The announced Maia's result is:

Theorem 3. *Let d be complete and T be d-continuous. In addition, assume that T is e-contractive and d is subordinated to e. Then, T is a Picard operator (modulo d).*

In particular, when $d = e$, Theorem 3 is just the Banach contraction principle [2]. However, its potential is much more spectacular; as certified by

Proposition 1. *Under these conventions, we have Theorem 3 $\Longrightarrow$ Theorem 2; hence (by the above) Maia's fixed point result implies Ran-Reurings'.*

Proof. Let $\alpha \in]0,1[$ be the number in (a02); and fix λ in $]1,1/\alpha[$. We claim that

$$e(x,y) := \sum_{n\geq 0} \lambda^n d(T^n x, T^n y) < \infty, \text{ for all } x,y \in X. \tag{2.1}$$

In fact, there exists from (b03), a (<>)-chain $\{z_1,...,z_k\}$ (for $k \geq 2$) in X with $z_1 = x$, $z_k = y$. By (b02), $T^n z_i <> T^n z_{i+1}$, $\forall n$, $\forall i \in \{1,...,k-1\}$; hence, via (a02), $d(T^n z_i, T^n z_{i+1}) \leq \alpha^n d(z_i, z_{i+1})$, for the same ranks (n,i). But then

$$d(T^n x, T^n y) \leq \sum_{i=1}^{k-1} d(T^n z_i, T^n z_{i+1}) \leq \alpha^n \sum_{i=1}^{k-1} d(z_i, z_{i+1}), \quad \forall n;$$

wherefrom (by the choice of λ)

$$\sum_{n\geq 0} \lambda^n d(T^n x, T^n y) \leq \sum_{n\geq 0} (\lambda\alpha)^n \sum_{i=1}^{k-1} d(z_i, z_{i+1}) < \infty;$$

hence the claim. The obtained map $e : X \times X \to R_+$ is *reflexive* $[e(x,x) = 0$, $\forall x \in X]$, *symmetric* $[e(y,y) = e(y,x)$, $\forall x,y \in X]$ and *triangular* $[e(x,z) \leq e(x,y) + e(y,z)$, $\forall x,y,z \in X]$. Moreover, in view of

$$e(x,y) = d(x,y) + \lambda e(Tx,Ty) \geq \lambda e(Tx,Ty), \ \forall x,y \in X,$$

d is subordinated to e. Note that e is *sufficient* in such a case $[e(x,y) = 0 \Longrightarrow x = y]$; hence, it is a (standard) metric on X. On the other hand, the same relation tells us that T is (e,μ)-contractive for $\mu = 1/\lambda \in]\alpha,1[$; hence, (by definition), e-contractive. This, along with the remaining conditions of Theorem 2, shows that Theorem 3 applies to these data; wherefrom, all is clear. □

3. Extensions of Maia's result

From these developments, it follows that Maia's result [15, Theorem 1] is an outstanding tool in the area; so, the question of enlarging it is of interest. A positive answer to this, in a convergence-metric setting, will be described below.

Let X be a nonempty set. Denote by $\mathcal{S}(X)$, the class of all sequences (x_n) in X. By a (sequential) *convergence structure* on X we mean, as in Kasahara [12], any part $\mathcal{C}$ of $\mathcal{S}(X) \times X$ with the properties

(c01) $x_n = x, \forall n \in N \Longrightarrow ((x_n);x) \in \mathcal{C}$
(c02) $((x_n);x) \in \mathcal{C} \Longrightarrow ((y_n);x) \in \mathcal{C}$, for each subsequence (y_n) of (x_n).

In this case, $((x_n);x) \in \mathcal{C}$ writes $x_n \xrightarrow{\mathcal{C}} x$; and reads: x is the $\mathcal{C}$-*limit* of (x_n). The set of all such x is denoted $\lim_n x_n$; when it is nonempty, we say that (x_n) is $\mathcal{C}$-*convergent*; and the class of all these will be denoted $\mathcal{S}_c(X)$. Assume that we fixed such an object, with

(c03) $\mathcal{C}$=separated: $\lim_n x_n$ is a singleton, for each (x_n) in $\mathcal{S}_c(X)$;

as usually, we shall write $\lim_n x_n = \{z\}$ as $\lim_n x_n = z$. (Note that, in the Fréchet terminology [6], this condition is automatically fulfilled, by the specific way of introducing the ambient convergence; see, for instance, Petruşel and Rus [20]). Let $(\leq)$ be a *quasi-order* (i.e.: reflexive and transitive relation) over X; and take a self-map T of X. The basic conditions to be imposed are

(c04) $X(T,\leq) := \{x \in X; x \leq Tx\}$ is nonempty
(c05) T is $\leq$-increasing $(x \leq y \Longrightarrow Tx \leq Ty)$.

We say that $x \in X(T,\leq)$ is a *Picard point* (modulo $(\mathcal{C},\leq,T)$) if **i)** $(T^n x; n \geq 0)$ is $\mathcal{C}$-convergent, **ii)** $z := \lim_n T^n x$ is in $\mathrm{Fix}(T)$ and $T^n x \leq z$, $\forall n$. If this happens for each $x \in X(T,\leq)$ and **iii)** $\mathrm{Fix}(T)$ is $(\leq)$-*singleton* $[z,w \in \mathrm{Fix}(T),\ z \leq w \Longrightarrow z = w]$, then T is called a *Picard operator* (modulo $(\mathcal{C},\leq)$). Note that, in this case, each $x^* \in \mathrm{Fix}(T)$ fulfills

$$\forall u \in X(T,\leq):\ x^* \leq u \Longrightarrow u \leq x^*; \tag{3.1}$$

i.e.: x^* is $(\leq)$-maximal in $X(T,\leq)$. In fact, assume that $x^* \leq u \in X(T,\leq)$. By i) and ii), $(T^n u; n \geq 0)$ $\mathcal{C}$-converges to some $u^* \in \mathrm{Fix}(T)$ with $T^n u \leq u^*$, $\forall n$; hence, $x^* \leq u \leq u^*$. Combining with iii) gives $x^* = u^*$; wherefrom $u \leq x^*$.

Concerning the sufficient conditions for such a property, an early statement of this type was established by Turinici [27]; cf. Section 5. Here, we propose a different approach, founded on ascending orbital concepts (in short: ao-concepts) and almost metrics. Some conventions are in order. Call the sequence $(z_n; n \geq 0)$ in X, *ascending* if $z_i \leq z_j$ for $i \leq j$; and T-*orbital* when $z_n = T^n x$, $n \geq 0$, for some $x \in X$; the intersection of these concepts is just the precise one. We say that **iv)** $(\leq)$ is $(ao,\mathcal{C})$-*self-closed* when the $\mathcal{C}$-limit of each $\mathcal{C}$-convergent ao-sequence is an upper bound of it, **v)** T is $(ao,\mathcal{C})$-*continuous* if $[(z_n)$=ao-sequence, $z_n \xrightarrow{\mathcal{C}} z$, $z_n \leq z, \forall n]$ implies $Tz_n \xrightarrow{\mathcal{C}} Tz$. Further, by an *almost metric* over X we shall mean any map $e: X \times X \to R_+$; supposed to be reflexive triangular and sufficient. This comes form the fact that such an object has all properties of a metric, excepting symmetry. Call the sequence (x_n), e-*Cauchy* when $[\forall \delta > 0,\ \exists n(\delta)\colon\ n(\delta) \leq n \leq m \Longrightarrow e(x_n,x_m) \leq \delta]$. We then say that **vi)** $(e,\mathcal{C})$ is *ao-complete*, provided [(for each ao-sequence) e-Cauchy $\Longrightarrow$ $\mathcal{C}$-convergent].

Let $\mathcal{F}(R_+)$ stand for the class of all functions $\varphi: R_+ \to R_+$. Denote by $\mathcal{F}_i(R_+)$, the subclass of all increasing $\varphi \in \mathcal{F}(R_+)$; and by $\mathcal{F}_1(R_+)$, the subclass of all $\varphi \in \mathcal{F}(R_+)$ with $\varphi(0) = 0$ and $[\varphi(t) < t,\ \forall t > 0]$. We shall term $\varphi \in \mathcal{F}(R_+)$, a *comparison function* if $\varphi \in \mathcal{F}_i(R_+) \cap \mathcal{F}_1(R_+)$ and $[\varphi^n(t) \to 0$, for all $t > 0]$. [Note that $\varphi \in \mathcal{F}_1(R_+)$ follows from $\varphi \in \mathcal{F}_i(R_+)$ and the last property; cf. Matkowski [16]; but, this is not essential for us]. A basic property of such functions (used in the sequel) is

$$(\forall \gamma > 0), (\exists \beta > 0), (\forall t):\ 0 \leq t < \gamma + \beta \Longrightarrow \varphi(t) \leq \gamma. \tag{3.2}$$

For completeness, we supply a proof of this, due to Jachymski [11]. Assume that the underlying property fails; i.e. (for some $\gamma > 0$):

$$\forall \beta > 0,\ \exists t \in [0, \gamma+\beta[, \text{ such that } \varphi(t) > \gamma \text{ (hence, } \gamma < t < \gamma + \beta).$$

As $\varphi \in \mathcal{F}_i(R_+)$, this yields $\varphi(t) > \gamma$, $\forall t > \gamma$. By induction, we get (for some $t > \gamma$) $\varphi^n(t) > \gamma$, $\forall n$; so (passing to limit as $n \to \infty$) $0 \geq \gamma$, contradiction.

Denote, for $x,y \in X$: $H(x,y) = \max\{e(x,Tx), e(y,Ty)\}$, $L(x,y) = \frac{1}{2}[e(x,Ty) + e(Tx,y)]$, $M(x,y) = \max\{e(x,y), H(x,y), L(x,y)\}$. Clearly,

$$M(x,Tx) = \max\{e(x,Tx), e(Tx,T^2x)\},\ \forall x \in X. \tag{3.3}$$

Call the self-map T, $(e,M;\leq;\varphi)$-*contractive* (for $\varphi \in \mathcal{F}(R_+)$), if

(c06) $e(Tx,Ty) \leq \varphi(M(x,y))$, $\forall x,y \in X$, $x \leq y$;

when this holds for at least one comparison function φ, the resulting convention reads: T is *extended* $(e,M;\leq)$-*contractive*.

Theorem 4. *Suppose that [in addition to (c04)+(c05)], T is extended $(e, M; \leq)$-contractive and $(ao, \mathcal{C})$-continuous, $(e, \mathcal{C})$ is ao-complete, and $(\leq)$ is $(ao, \mathcal{C})$-self-closed. Then, T is a Picard operator (modulo $(\mathcal{C}, \leq)$).*

Proof. Let $x^*, u^* \in \mathrm{Fix}(T)$ be such that $x^* \leq u^*$. By the contractive condition, $e(x^*, u^*) = 0$; wherefrom, $x^* = u^*$; and so, $\mathrm{Fix}(T)$ is $(\leq)$-singleton. It remains to show that each $x = x_0 \in X(T, \leq)$ is a Picard point (modulo $(\mathcal{C}, \leq, T)$). Put $x_n = T^n x$, $n \geq 0$; and let $\varphi \in \mathcal{F}(R_+)$ be the comparison function given by the extended $(e, M; \leq)$-contractivity of T.

I) By the contractive condition and (3.3),

$$e(x_{n+1}, x_{n+2}) \leq \varphi(M(x_n, x_{n+1})) = \varphi[\max\{e(x_n, x_{n+1}), e(x_{n+1}, x_{n+2})\}], \forall n.$$

If (for some n) the maximum in the right hand side is $e(x_{n+1}, x_{n+2})$, then (via $\varphi \in \mathcal{F}_1(R_+)$) $e(x_{n+1}, x_{n+2}) = 0$; so that (as e=sufficient) $x_{n+1} \in \mathrm{Fix}(T)$; and we are done. Suppose that this alternative fails: $e(x_{n+1}, x_{n+2}) \leq \varphi(e(x_n, x_{n+1}))$, for all n. This yields (by an ordinary induction) $e(x_n, x_{n+1}) \leq \varphi^n(e(x_0, x_1))$, $\forall n$; wherefrom $e(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

II) We claim that $(x_n; n \geq 0)$ is e-Cauchy in X. Denote, for simplicity, $E(k, n) = e(x_k, x_{k+n})$, $k, n \geq 0$. Let $\gamma > 0$ be arbitrary fixed; and $\beta > 0$ be the number appearing in (3.2); without loss, one may assume that $\beta < \gamma$. By the preceding step, there exists a rank $m = m(\beta)$ such that

$$k \geq m \text{ implies } E(k, 1) < \beta/2 < \beta < \gamma. \tag{3.4}$$

The desired property follows from the inductive type relation

$$\forall n \geq 0\text{:} \;\; [E(k, n) < \gamma + \beta/2, \text{ for each } k \geq m]. \tag{3.5}$$

The case $n = 0$ is trivial; while the case $n = 1$ is clear, via (3.4). Assume that (3.5) is true, for all $n \in \{1, ..., p\}$ (where $p \geq 1$); we want to establish that it holds as well for $n = p+1$. So, let $k \geq m$ be arbitrary fixed. By the induction hypothesis and (3.4), $e(x_k, x_{k+p}) = E(k, p) < \gamma + \beta/2$ and $H(x_k, x_{k+p}) = \max\{E(k, 1), E(k+p, 1)\} < \beta/2$. Moreover, the same premises give (by the triangular property)

$$\begin{aligned} &L(x_k, x_{k+p}) = (1/2)[E(k, p+1) + E(k+1, p-1)] \leq \\ &(1/2)[E(k, p) + E(k+p, 1) + E(k+1, p-1)] < \gamma + \beta; \end{aligned}$$

wherefrom $M(x_k, x_{k+p}) < \gamma + \beta$; so, by the contractive condition and (3.2),

$$E(k+1, p) = e(x_{k+1}, x_{k+p+1}) = e(Tx_k, Tx_{k+p}) \leq \varphi(M(x_k, x_{k+p})) \leq \gamma;$$

which "improves" the previous evaluation (3.5) of our quantity. This, along with (3.4) and the triangular property, gives $E(k, p+1) = e(x_k, x_{k+p+1}) < \gamma + \beta/2$.

III) As $(e, \mathcal{C})$ is ao-complete, (3.5) tells us that $x_n \xrightarrow{\mathcal{C}} x^*$ for some $x^* \in X$. Moreover, as $(\leq)$ is $(ao, \mathcal{C})$-self-closed, we have $x_n \leq x^*$, $\forall n$; hence, in particular, $x \leq x^*$. Combining with the $(ao, \mathcal{C})$-continuity of T, yields $x_{n+1} = Tx_n \xrightarrow{\mathcal{C}} Tx^*$; wherefrom (as $\mathcal{C}$ is separated), $x^* \in \mathrm{Fix}(T)$. □

Now letting d be a metric on X, the associated convergence $\mathcal{C} := (\xrightarrow{d})$ is separated; moreover, the ao-complete property of $(e, \mathcal{C})$ is holding whenever d is complete and subordinated to e. Clearly, this last property is trivially assured if $d = e$; when Theorem 4 is comparable with the main result in Agarwal, El-Gebeily and O'Regan [1]. In fact, a little modification of the working hypotheses allows us getting the whole conclusion of the quoted statement; we do not give details.

4. Particular aspects

Let X be a nonempty set; and $T : X \to X$ be a self-map of X. Further, take a separated (sequential) convergence structure $\mathcal{C}$ on X.

(A) Let $e(.,.)$ be an almost metric over X. A basic particular case of the previous developments corresponds to $(\leq) = X \times X$ (=the *trivial* quasi-order on X). Then, (c04)+(c05) are holding; and the resulting Picard concept becomes a Picard property (modulo $\mathcal{C}$) of T, which writes: **i)** Fix(T) is a singleton, $\{x^*\}$, **ii)** $T^n x \xrightarrow{\mathcal{C}} x^*$, for each $x \in X$. Moreover, the $(ao, \mathcal{C})$-self-closedness of $(\leq)$ is fulfilled; and the remaining ao-concepts become orbital concepts (in short: o-concepts). Precisely, call T, $(o, \mathcal{C})$*-continuous* if [(z_n)=o-sequence, $z_n \xrightarrow{\mathcal{C}} z$ imply $Tz_n \xrightarrow{\mathcal{C}} Tz$]; likewise, call $(e, \mathcal{C})$, *o-complete* if [(for each o-sequence) e-Cauchy $\Longrightarrow$ $\mathcal{C}$-convergent]. Finally, concerning (c06), let us say that T is $(e; M; \varphi)$*-contractive* (where $\varphi \in \mathcal{F}(R_+)$), provided

(d01) $e(Tx, Ty) \leq \varphi(M(x, y))$, $\forall x, y \in X$;

if this holds for at least one comparison function φ, the resulting convention reads: T is *extended* (e, M)*-contractive.* Putting these together, one gets the following version of Theorem 4:

Corollary 1. *Suppose that T is extended e-contractive and $(o, \mathcal{C})$-continuous, and $(e, \mathcal{C})$ is o-complete. Then, T is a Picard operator (modulo $\mathcal{C}$).*

The obtained statement includes Kasahara's fixed point principle [12], when e is a metric on X. On the other hand, if d is a metric on X and $\mathcal{C} := (\xrightarrow{d})$, the o-complete property of $(e, \mathcal{C})$ is assured when d is complete and subordinated to e. This, under a linear choice of the comparison function ($\varphi(t) = \alpha t$, $t \in R_+$, for $0 < \alpha < 1$), tells us that Corollary 1 includes Theorem 3. Finally, when $d = e$, Corollary 1 reduces to Jachymski's result [10].

(B) An interesting version of Corollary 1 was provided in the 2008 paper by O'Regan and Petruşel [19, Theorem 3.3]. Let $(X, T, \mathcal{C})$ be endowed with their precise general meaning; and $d(.,.)$ be a (standard) metric on X. As before, we are interested to give sufficient conditions under which T be a Picard operator (modulo $\mathcal{C}$). Take an *order* $(\leq)$ on X; and put $X_{(\leq)} = (\leq) \cup (\geq)$, where $(\geq)$ stands for the *dual* order. This subset is just the graph of the relation $<>$ over X introduced as in (a01); so, it may be identified with the underlying relation. As a consequence, $X_{(\leq)}$ is *reflexive* [$(x, x) \in X_{(\leq)}$, for each $x \in X$] and *symmetric* [$(x, y) \in X_{(\leq)}$ iff $(y, x) \in X_{(\leq)}$]; but not in general transitive, as simple examples show. Further, let us say that $(d, \mathcal{C})$ is *o-complete* if [(for each sequence) d-Cauchy $\Longrightarrow$ $\mathcal{C}$-convergent]. Finally, call T, $(d, \leq; \varphi)$*-contractive* (for $\varphi \in \mathcal{F}(R_+)$), if

(d02) $d(Tx, Ty) \leq \varphi(d(x, y))$, $\forall x, y \in X$, $x \leq y$;

when this holds for at least one comparison function φ, the resulting convention reads: T is $(d, \leq)$*-contractive.*

Corollary 2. *Assume that (a03)+(b02) hold, T is $(d, \leq)$-contractive and $(o, \mathcal{C})$-continuous, $(d, \mathcal{C})$ is complete, and*

(d03) $(x, y), (y, z) \in X_{(\leq)} \Rightarrow (x, z) \in X_{(\leq)}$ *(i.e.: $X_{(\leq)}$ is transitive)*

(d04) $(x, y) \notin X_{(\leq)} \Rightarrow \exists\ c = c(x, y) \in X$*:* $(x, c), (y, c) \in X_{(\leq)}$.

Then, T is a Picard map (modulo $\mathcal{C}$).

Proof. We claim that Corollary 1 is applicable to such data. This will follow from

$$X \times X = X_{(\le)} \text{ (i.e.: the ambient order } (\le) \text{ is linear).}$$

In fact, let $x, y \in X$ be arbitrary fixed. If $(x, y) \in X_{(\le)}$, we are done; so, assume that $(x, y) \notin X_{(\le)}$. By (d04), there exists $c = c(x, y) \in X$ such that $(x, c) \in X_{(\le)}$, $(y, c) \in X_{(\le)}$. This, along with the symmetry of $X_{(\le)}$, gives $(c, y) \in X_{(\le)}$; hence, by (d03), $(x, y) \in X_{(\le)}$. As a consequence, the $(d, \le; \varphi)$-contractive property for T is to be written in the "amorphous" form of (d01) [with $d(.,.)$ in place of $e(.,.)$ and $M(.,.)$]; wherefrom, all is clear. Note that, from such a perspective, conditions (a03)+(b02) are superfluous. □

5. Old approach (1986)

In the following, a summary of the 1986 results in Turinici [27] is being sketched, for completeness reasons.

(A) Let (X, d) be a complete metric space and T be a self-map of X. Assume that for each $x \in X$ there exists a $n(x) \in N_0 := N \setminus \{0\}$ such that $T^{n(x)}$ is (metrically) contractive at x; then, we may ask of under which additional conditions is T endowed with a Picard property (cf. Section 1). A first answer to this question was given, in the continuous case, by Sehgal [24] through a specific iterative procedure; a reformulation of it for discontinuous maps was performed in Guseman's paper [7]. During the last decade, some technical extensions – involving the contractive condition – of these results were obtained by Ciric [3], Khazanchi [13], Iseki [9], Rhoades [22] and Singh [25]. The most general statement of this kind, obtained by Matkowski [16], reads as follows. For each $m \in N_0$, let $\mathcal{F}(R_+^m)$ stand for the class of all functions $f : R_+^m \to R_+$; and $\mathcal{F}_i(R_+^m)$ the subclass of all $f \in \mathcal{F}(R_+^m)$, increasing in each variable. The iterative contraction property below is considered:

(e01) $\exists f \in \mathcal{F}_i(R_+^5)$ such that: $\forall x \in X$, $\exists n(x) \in N_0$ with $d(T^{n(x)}x, T^{n(x)}y) \le f(d(x, T^{n(x)}x), d(x, y), d(x, T^{n(x)}y), d(T^{n(x)}x, y), d(T^{n(x)}y, y))$, $\forall y \in X$.

Given $f \in \mathcal{F}_i(R_+^5)$ like before, denote $g(t) = f(t, t, t, 2t, 2t)$, $t \ge 0$; clearly, it is an element of $\mathcal{F}_i(R_+)$. We shall say that f is *normal* provided

(e02) $g \in \mathcal{F}_1(R_+)$ and $[t - g(t) \to \infty$ as $t \to \infty]$

(e03) $\lim_n g^n(t) = 0$, for each $t > 0$.

(As already remarked, (e03) implies the first part of (e02), under the properties of g; we do not give details).

Theorem 5. *Suppose that there exists a normal function $f \in \mathcal{F}_i(R_+^5)$ in such a way that (e01) holds. Then, T is a Picard operator (modulo d).*

A direct examination of the above conditions shows that, by virtue of

$$d(T^{n(x)}x, y) \le d(x, T^{n(x)}x) + d(x, y),\ \ x, y \in X,$$

$$d(T^{n(x)}y, y) \le d(x, T^{n(x)}y) + d(x, y),\ \ x, y \in X,$$

a slight extension of Theorem 5 might be reached if one replaces (e01) by

(e04) $d(T^{n(x)}x, T^{n(x)}y) \le F(d(x, T^{n(x)}x), d(x, y), d(x, T^{n(x)}y))$, $y \in X$,

where $F : R_+^3 \to R_+$ is defined as

$$F(\xi, \eta, \zeta) = f(\xi, \eta, \zeta, \xi + \eta, \zeta + \eta),\ \ \xi, \eta, \zeta \in R_+.$$

A natural question to be solved is that of determining what happens when the right-hand side of (e04) depends on the (abstract) variable $x \in X$ and the (real) variables $((d(x, T^i x); 1 \le i \le n(x)),\ (d(x, T^j y); 0 \le j \le n(x))$; or, in other words, when the function $F = F(x)$ acts from $R_+^{2n(x)+1}$ to R_+. At the same time, observe that, from a "relational" viewpoint, the result we just recorded may be deemed as being expressed modulo the *trivial* quasi-ordering on X; so that, a formulation of it in terms of genuine quasi-orderings would be of interest. It is precisely our main aim to get a generalization – under the above lines – of Theorem 5.

(B) Let (X, d) be a metric space and $\le$ be a quasi-ordering (i.e.: reflexive and transitive relation) over X. A sequence $(x_n; n \in N)$ in X will be said to be *increasing* when $x_i \le x_j$ for $i \le j$. Take the self-map T of X according to

(e05) $Y := \{x \in X; x \le Tx\}$ is not empty
(e06) T is increasing ($x \le y$ implies $Tx \le Ty$).

In addition, the specific condition will be accepted:

(e07) for each x in Y there exist $n(x) \in N_0$, $f(x) \in \mathcal{F}_i(R_+^{2n(x)+1})$, with $d(T^{n(x)}x, T^{n(x)}y) \le f(x)(d(x, Tx), ..., d(x, T^{n(x)}x); d(x, y), ..., d(x, T^{n(x)}y))$, for all $y \in Y$ with $x \le y$.

For the arbitrary fixed $x \in Y$, let $g(x)$ indicate the element of $\mathcal{F}_i(R_+)$, given as $g(x)(t) = f(x)(t, ..., t; t, ..., t)$, $t \ge 0$. We shall say that the family (of (e07)) $((n(x), f(x)); x \in Y)$ is *iterative T-normal* provided, for each $x_0 \in Y$,

(e08) $g(x_0) \in \mathcal{F}_1(R_+)$ and $t - g(x_0)(t) \to \infty$ as $t \to \infty$,
(e09) $\lim_k g(x_k) \circ ... \circ g(x_0)(t) = 0$, $t > 0$, where $[n_0 = n(x_0),\ x_1 = T^{n_0}x_0]$ and, inductively, $[n_i = n(x_i),\ x_{i+1} = T^{n_i}x_i]$, $i \ge 1$.

The following auxiliary fact will be useful.

Proposition 2. *Let (e05)-(e07) hold; and the family $((n(x), f(x)); x \in Y)$ [attached to (e07)] be iterative T-normal. Then, the following conclusions hold*
i) *for each $x \in Y$, $(T^m x; m \in N)$ is increasing Cauchy (in X)*
ii) *$d(T^m x, T^m y) \to 0$ as $m \to \infty$, for all $y \in Y$, $x \le y$.*

Proof. Let $x \in Y$ be given. We firstly claim that

$$d(x, T^m x) \le t,\ m \in N, \text{ for some } t = t(x) > 0. \tag{5.1}$$

Indeed, it follows by (e08) that, given $\alpha > 0$, there exists $\beta = \beta(\alpha, x) > \alpha$ with

$$t \le \alpha + g(x)(t) \text{ implies } t \le \beta. \tag{5.2}$$

Put $\alpha = \max\{d(x, Tx), ..., d(x, T^{n(x)}x)\}$. We claim that (5.1) holds with $t = \beta$. In fact, suppose that the considered assertion would be false; and let m denote the infimum of those ranks for which the reverse of (5.1) takes place. Clearly, $m > n(x)$, $d(x, T^k x) \le \beta$, $k \in \{1, ..., m-1\}$, and $d(x, T^m x) > \beta$; so that, by (e07),

$$\begin{aligned} &d(x, T^m x) \le d(x, T^{n(x)}x) + d(T^{n(x)}x, T^m x) \le \\ &\alpha + f(x)(d(x, Tx), ..., d(x, T^{n(x)}x); d(x, T^{m-n(x)}x), ..., d(x, T^m x)) \le \\ &\alpha + f(x)(\alpha, ..., \alpha; \beta, ..., \beta, d(x, T^m x)) \le \alpha + g(x)(d(x, T^m x)); \end{aligned}$$

contradicting (5.2) and proving our assertion. In this case, letting $x = x_0$ in Y, put $n_0 = n(x_0)$, $m_0 = n_0$, $x_1 = T^{n_0}x_0 = T^{m_0}x_0$ and, inductively,

$$n_i = n(x_i), m_i = n_0 + ... + n_i, x_{i+1} = T^{n_i}x_i = T^{m_i}x_0,\ i \ge 1.$$

By (5.1), $d(x_0, T^m x_0) \le t_0$, $m \in N$, for some $t_0 > 0$; so combining with (e07):

$$d(x_1, T^m x_1) = d(T^{n_0} x_0, T^{n_0} T^m x_0) \le f(x_0)(d(x_0, Tx_0), ..., d(x_0, T^{n_0} x_0);$$
$$d(x_0, T^m x_0), ..., d(x_0, T^{n_0+m} x_0)) \le g(x_0)(t_0), m \in N;$$

or equivalently, $d(T^{m_0} x_0, T^m x_0) \le g(x_0)(t_0)$, $m \ge m_0$. Again via (e07),

$$d(x_2, T^m x_2) = d(T^{n_1} x_1, T^{n_1} T^m x_1) \le f(x_1)(d(x_1, Tx_1), ..., d(x_1, T^{n_1} x_1);$$
$$d(x_1, T^m x_1), ..., d(x_1, T^{n_1+m} x_1)) \le g(x_1) \circ g(x_0)(t_0), m \in N;$$

or equivalently: $d(T^{m_1} x_0, T^m x_0) \le g(x_1) \circ g(x_0)(t_0)$, $m \ge m_1$; and so on. By a finite induction procedure one gets $d(x_{k+1}, T^m x_{k+1}) \le g(x_k) \circ ... \circ g(x_0)(t_0), m, k \in N$; or equivalently (for each $k \in N$)

$$d(T^{m_k} x_0, T^m x_0) \le g(x_k) \circ ... \circ g(x_0)(t_0),\ m \ge m_k;$$

wherefrom, taking (e09) into account, $(T^n x_0; n \in N)$ is an increasing Cauchy sequence. Finally, given $y_0 \in Y$ with $x_0 \le y_0$, put $y_1 = T^{n_0} y_0$ and, inductively, $y_{i+1} = T^{n_i} y_i = T^{m_i} y_0$, $i \ge 1$. Again by (5.1),

$$d(x_0, T^m x_0), d(x_0, T^m y_0) \le t_0,\ m \in N, \text{ for some } t_0 > 0.$$

This fact, combined with (e07), leads us, by the same procedure as before, at

$$d(x_{k+1}, T^m x_{k+1}), d(x_{k+1}, T^m y_{k+1}) \le g(x_k) \circ ... \circ g(x_0)(t_0),\ m, k \in N;$$

or equivalently (for each $k \in N$)

$$d(T^{m_k} x_0, T^m x_0), d(T^{m_k} x_0, T^m y_0) \le g(x_k) \circ ... \circ g(x_0)(t_0),\ m \ge m_k;$$

proving the desired conclusion and completing the argument. □

(C) Let X, d and $\le$ be endowed with their previous meaning. Given the sequence $(x_n; n \in N)$ in X and the point $x \in X$, define $x_n \uparrow x$ as: $(x_n; n \in N)$ is increasing and convergent to x. Term the triplet $(X, d; \le)$, *quasi-order complete*, provided each increasing Cauchy sequence converges. Note that any complete metric space is quasi-order complete; but the converse is not in general valid. Further, given the self-map T of X, call it *continuous at the left* when $x_n \uparrow x$ and $x_n \le x$, $n \in N$, imply $Tx_n \to Tx$. Also, the ambient quasi-ordering $\le$ will be said to be *self-closed* when $x \le y_n$, $n \in N$ and $y_n \uparrow y$ imply $x \le y$; note that any semi-closed quasi-ordering in Nachbin's sense [17, Appendix] is necessarily self-closed. The first main result of the present note is

Theorem 6. *Let the conditions of Proposition 2 be fulfilled; and (in addition) $(X, d; \le)$ is quasi-order complete, $\le$ is self-closed, and T is continuous at the left. Then, the following conclusions will be valid*

iii) *$Z := \{x \in X; x = Tx\}$ is not empty*

iv) *for every $x \in Y$, $(T^n x; n \in N)$ converges to an element of Z*

v) *if $x, y \in Y$ are comparable, $(T^n x; n \in N)$ and $(T^n y; n \in N)$ have the same limit (in Z).*

Proof. By Proposition 2 and the quasi-order completeness of $(X, d; \le)$, it follows that, for the arbitrary fixed $x \in Y$, $T^n x \uparrow z$ for some $x \in X$. As $\le$ is self-closed, $T^n x \le z$, $n \in N$; so that, combining with the left continuity of T one gets $T^n x \uparrow Tz$; hence $z = Tz$. The proof is thereby complete. □

Now, it is natural to ask of what happens when T is no longer continuous at the left. Some conventions are in order. Call $\le$, *anti self-closed* when $y_n \le x$, $n \in N$, and $y_n \uparrow y$ imply $y \le x$; observe at this moment that a sufficient condition for $\le$ to be anti self-closed is that $\ge$ (its dual) be semi-closed. Further, call $\le$, *interval closed* when it is both self-closed and anti self-closed. Our second main result is

Theorem 7. *Let the conditions of Proposition 2 be fulfilled; and (in addition) $(X, d; \le)$ is quasi-order complete and $\le$ is an interval closed ordering. Then, conclusions* **iii)**- **v)** *of Theorem 6 continue to hold; and, moreover,*

vi) *for each $x \in Y$ the element $z = \lim_n T^n x$ in Z has the properties (a) $x \le z$, (b) $z \le y \in Y$ implies $z = y$.*

Proof. Let $x \in Y$ be arbitrary fixed. By Proposition 2, $T^n x \uparrow z$, for some $z \in X$. hence (as $\le$ is self-closed), $x \le T^n x \le z$, $n \in N$. It immediately follows that $T^n x \le Tz$, $n \in N$; so (by the anti self-closedness of $\le$), $z \in Y$. Now, $x \le z \in Y$ gives, again by Proposition 2, $T^n z \uparrow z$ (hence $Tz \le T^n z \le z$, $n \in N_0$) and therefore (as $\le$ is ordering) $z \in Z$. The remaining part is evident. □

It remains now to discuss the alternative:

(T is not continuous at the left) and ($\le$ is not an interval closed ordering).

To this end, assume that, for any $x \in Y$, the function $f(x) \in \mathcal{F}_i(R_+^{2n(x)+1})$ given by (e07) fulfills

(e10) for each $(\alpha_1, .., \alpha_{n(x)}) \in R_+^{n(x)}$ with $\alpha_{n(x)} > 0$ there exists $\beta > 0$ with $\beta + f(x)(\alpha_1, ..., \alpha_{n(x)}; \beta, ..., \beta) < \alpha_{n(x)}$

(e11) for each $(\alpha_1, ..., \alpha_{n(x)}) \in R_+^{n(x)}$ with $\alpha_1 > 0$, $\alpha_{n(x)} = 0$, we have $f(x)(\alpha_1, ..., \alpha_{n(x)}; \alpha_1, ..., \alpha_{n(x)}, \alpha_1) < \alpha_1$.

Now, as a completion of the above results, we have

Theorem 8. *Let the conditions of Proposition 2 be fulfilled; and (in addition) $(X, d; \le)$ is quasi-order complete, (e10)+(e11) hold, and $\le$ is an interval closed quasi-ordering. Then, conclusions* **iii)**–**vi)** *of Theorem 7 still remain valid.*

Proof. Let $x \in Y$ be arbitrary fixed. By the above reasoning, $T^n x \uparrow z$, for some $z \in Y$; with, in addition (cf. Proposition 2): $[x \le T^n x \le z, n \in N]$ and $[T^n z \uparrow z]$. Assume that $z \ne T^{n(z)} z$; and let $\beta > 0$ be the number attached (via (e10)) to $\alpha_1 := d(z, Tz)$,..., $\alpha_{n(x)} := d(z, T^{n(z)} z)$. By the convergence property above, there exists $k(\beta) \in N$ such that $d(z, T^k z) \le \beta$, $\forall k \ge k(\beta)$; and this gives for all ranks $m \ge k(\beta) + n(z)$,

$$\begin{aligned}
&d(z, T^{n(z)} z) \le d(z, T^m z) + d(T^{n(z)} z, T^m z) \le d(z, T^m z) + \\
&f(z)(d(z, Tz), ..., d(z, T^{n(z)} z); d(z, T^{m-n(z)} z, ..., d(z, T^m z)) \le \\
&\beta + f(z)(d(z, Tz), ..., d(z, T^{n(z)} z); \beta, ..., \beta) < d(z, T^{n(z)} z);
\end{aligned}$$

contradiction; hence $z = T^{n(z)} z$. Moreover,

$$\begin{aligned}
&d(z, Tz) = d(T^{n(z)} z, T^{n(z)} Tz) \le \\
&f(z)(d(z, Tz), ..., d(z, T^{n(z)} z); d(z, Tz), ..., d(z, T^{n(z)} z), d(z, T^{n(z)} Tz)) = \\
&f(z)(d(z, Tz), ..., d(z, T^{n(z)-1} z), 0; d(z, Tz), ..., d(z, T^{n(z)-1} z), 0, d(z, Tz));
\end{aligned}$$

wherefrom, if $z \ne Tz$, (e11) will be contradicted. Hence the conclusion. □

Some remarks are in order. Theorem 6 may he viewed as a quasi-order extension of Sehgal's result we just quoted (cf. also Dugundji and Granas [5, Ch 1, Sect 3]) while Theorem 7 is a quasi-order "functional" version of Matkowski's contribution (Theorem 5). At the same time, Theorem 8 - although formulated as a fixed point result - may be deemed in fact as a maximality principle in $(Y, \leq)$; so, it is comparable under this perspective with a related author's one [26] obtained by means of a "compactness" procedure like in Krasnoselskii and Sobolev [14].

(D) *Note added in 2011*

From these developments, the following statement is deductible. Let the quasi-ordered metric space $(X, d, \leq)$ the self-map T of X be taken as in (e05)+(e06). In addition, the specific condition will be accepted:

(e12) there exists $f \in \mathcal{F}_i(R_+)$ such that: for each x in Y there exists $n(x) \in N_0$ with $d(T^{n(x)}x, T^{n(x)}y) \leq f(d(x, y))$, for all $y \in Y$ with $x \leq y$.

Note that, in such a case, the iterative normality of $((n(x); f); x \in Y)$ is characterized by (e02)+(e03), with f in place of g; and referred to as: f is *normal* (see above). From Theorem 6 we then get, formally

Theorem 9. *In addition to (e05)+(e06), assume that the function f (appearing in (e12)) is normal, $(X, d; \leq)$ is quasi-order complete, $\leq$ is self-closed, and T is continuous at the left. Then, conclusions* **iii)**–**v)** *of Theorem 6 are retainable.*

In particular, any linear comparison function f (in the sense: $f(t) = \alpha t$, $t \in R_+$, for $0 < \alpha < 1$) is normal. Then, Theorem 9 includes the essential conclusions of the Ran-Reurings result (Theorem 1). [In fact, under appropriate conditions, it may give us all conclusions in the quoted statement; we do not give details]. Note that Theorem 9 is not yet covered by the existing fixed point statements in the realm of quasi-ordered metric spaces. Further aspects will be delineated elsewhere.

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